# A note on Stanley's conjecture for monomial ideals

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**Abstract**

We prove that the Stanley's conjecture holds for monomial ideals $I \subset K[x_1,\ldots,x_n]$ generated by at most $2n-1$ monomials, i.e. $\operatorname{sdepth}(I) \geq \operatorname{depth}(I)$.

**Keywords:** Stanley depth, monomial ideal.

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# Introduction

Let $K$ be a field and $S = K[x_1,\ldots,x_n]$ the polynomial ring over $K$. Let $M$ be a $\mathbb{Z}^n$-graded $S$-module. A *Stanley decomposition* of $M$ is a direct sum $\mathcal{D} : M = \bigoplus_{i=1}^r m_i K[Z_i]$ as $K$-vector space, where $m_i \in M$, $Z_i \subset \{x_1,\ldots,x_n\}$ such that $m_iK[Z_i]$ is a free $K[Z_i]$-module. We define $\operatorname{sdepth}(\mathcal{D}) = min_{i=1}^r|Z_i|$ and $\operatorname{sdepth}(M) = max\{\operatorname{sdepth}(M)|\ \mathcal{D}$ is a Stanley decomposition of $M\}$. The number $\operatorname{sdepth}(M)$ is called the *Stanley depth* of $M$. Herzog, Vladoiu and Zheng show in [9] that this invariant can be computed in a finite number of steps if $M = I/J$, where $J \subset I \subset S$ are monomial ideals.

There are two important particular cases. If $I \subset S$ is a monomial ideal, we are interested in computing $\operatorname{sdepth}(S/I)$ and $\operatorname{sdepth}(I)$. There are some papers regarding this problem, like [9],[14],[2],[11], [16] and [6]. Stanley's conjecture says $\operatorname{sdepth}(M) \geq \operatorname{depth}(M)$, where $M$ is a finitely generated multigraded $S$-module. The Stanley conjecture for $S/I$ and $I$ was proved in some special cases, but it remains open in the general case. See for instance, [4], [8], [10], [1], [3] and [13].

In this paper, we prove that Stanley's conjecture holds for monomial ideals $I \subset S = K[x_1,\ldots,x_n]$ generated by $\leq 2n-1$ monomials, see Theorem 1.5. In order to prove this, the key result is Theorem 1.4. Proposition 1.6 suggests that Theorem 1.5 can be extended for monomial ideals generated by $2n$ or $2n+1$ monomials.

# 1 Main results

First, we recall the following results.

**Theorem 1.1.** *[12, Theorem 2.1] Let $I \subset S$ be a monomial ideal (minimally) generated by $m$ monomials. Then*

$$\operatorname{sdepth}(I) \geq \max\{1, n - \left\lfloor \frac{m}{2} \right\rfloor\}.$$

**Proposition 1.2.** *[15, Corrolary 1.3] Let $I \subset S$ be a monomial ideal. Then* $\operatorname{depth}(I : v) \geq \operatorname{depth}(I)$ *for all monomials $v \notin I$.*

If $u \in S$ is a monomial, we denote $supp(u) = \{x_j :\ x_j|u\}$.

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**Lemma 1.3.** *Let $I = (v_1, \dots, v_m) \subset S$ be a monomial ideal which is not principal, where $v_1, \dots, v_m \in S$ are some monomials, $n \geq 2$ and $2 \leq m < 2n$. Denote $t_j = |\{i : x_j | v_i\}|$. Then, there exists a $j \in [n]$ such that $t_j \geq m - 2n + 3$.*

*Proof.* If $m \leq n$ there is nothing to prove, since $m-2n+3 \leq 3-n \leq 1$. If $n < m < 2n$, we have $m-2n+3 \leq 2n-1-2n+3 = 2$. On the other hand, since $\sum_{j=1}^{n} t_j = \sum_{i=1}^{m} |supp(v_i)| \geq m > n$, it follows that there exists a $j \in [n]$ with $t_j \geq 2$. $\square$

**Theorem 1.4.** *Let $I = (v_1, \dots, v_m) \subset S$ be a monomial ideal which is not principal, where $v_1, \dots, v_m \in S$ are some monomials, $n \geq 2$ and $2 \leq m < 2n$. Denote $t_j = |\{i : x_j | v_i\}|$. If $\operatorname{depth}(I) \geq n - k$ then, there exists a $j \in [n]$ such that $t_j \geq m - 2k + 1$.*

*Proof.* We use induction on $n$ and $n - k \geq 1$. If $n - k = 1$, since $m - 2k + 1 = m - 2n + 3$, we are done by the previous lemma. If $n = 2$, since $I$ is not principal, it follows that $k = 1$ and, as before, we are done. Now, assume $n \geq 3$ and $n - k > 1$.

If there exists a $j \in [n]$ with $t_j \geq m - 2k + 1$ we are done. Suppose this is not the case. We may also assume that there exists a variable, let's say $x_n$, such that $x_n \notin \sqrt{I}$, otherwise $I$ would be Artinian. We consider the ideal $I' := (I : x_n^\infty) \cap S'$, where $S' = K[x_1, \dots, x_{n-1}]$. Obviously, $I' = (v'_1, \dots, v'_m)$, where $v'_i := v_i / x_n^{\alpha_i}$ and $\alpha_i = deg_{x_n}(v_i) := \max\{k : x_n^k | v_i\}$.

If $I'$ is principal, we can assume that $I' = (v'_1)$. Therefore, if $x_j \in supp(v'_1)$, it follows that $t_j = m$. If $I'$ is not principal, by 1.2, it follows that $\operatorname{depth}_{S'}(I') \geq \operatorname{depth}_S(I) - 1 = (n-1) - k$. For all $j \in [n-1]$, we denote $t'_j = |\{i : x_j | v'_m\}|$. Note that $t_j = t'_j$ for all $j \in [n-1]$. Using the induction hypothesis, it follows that there exists a $j \in [n-1]$ such that $t_j = t'_j \geq m - 2k + 1$. $\square$

**Theorem 1.5.** *Let $I \subset S$ be a monomial ideal with at most $2n - 1$ monomial generators. Then $\operatorname{sdepth}(I) \geq \operatorname{depth}(I)$.*

*Proof.* If $I$ is principal there is nothing to prove. Assume $I$ is not principal and $\operatorname{depth}(I) = n - k$. We denote $G(I) = \{v_1, \dots, v_m\}$, $m := |G(I)|$ and set $\epsilon(I) := \sum_{i=1}^{m} deg(v_i)$. We use induction on $\epsilon(I)$. If $\epsilon(I) = m$, then $I$ is generated by variables, and therefore, by [5, Theorem 2.2] and [9, Lemma 3.6], we have $\operatorname{sdepth}(I) = n - \left\lfloor \frac{m}{2} \right\rfloor \geq n - m + 1 = \operatorname{depth}(I)$.

According to 1.4, we can assume that $r := t_n(I) \geq m - 2k + 1$. If $r = m$, then $I = x_n(I : x_n)$ and by induction hypothesis and 1.2, we get $\operatorname{sdepth}(I) = \operatorname{sdepth}((I : x_n)) \geq \operatorname{depth}(I : x_n) \geq \operatorname{depth}(I)$.

By reordering the generators of $I$, we may assume that $x_n | v_1, \dots, x_n | v_r$ and $x_n$ does not divide $v_{r+1}, \dots, v_m$. Let $S' = K[x_1, \dots, x_{n-1}]$. We write:

$$I = (I \cap S') \oplus x_n(I : x_n).$$

Note that $I \cap S' = (v_{r+1}, \dots, v_m) \cap S'$. By 1.1 it follows that:

$$\operatorname{sdepth}(I \cap S') \geq n-1-\left\lfloor \frac{m-r}{2} \right\rfloor \geq n-1-\left\lfloor \frac{m-(m-2k+1)}{2} \right\rfloor \geq n-1-(k-1) = \operatorname{depth}(I).$$

By induction hypothesis and 1.2 we have $\operatorname{sdepth}((I : x_n)) \geq \operatorname{depth}(I : x_n) \geq \operatorname{depth}(I)$. Finally, we obtain a Stanley decomposition of $I$ with it's Stanley depth $\geq \operatorname{depth}(I)$. $\square$

Let $I \subset S$ be a monomial ideal. We assume that $G(I) = \{v_1, \ldots, v_m\}$, where $G(I)$ is the set of minimal monomial generators of $I$. We denote $g(I) = |G(I)|$, the number of minimal generators of $I$. For $j \in [n]$, we denote $t_j(I) = |\{u \in G(I) : \ x_j|u\}|$.

Considering the following proposition, which generalize Lemma 1.3, it would be expected to extend Theorem 1.5 for monomial ideals minimally generated by $2n$ or $2n+1$ monomials.

**Proposition 1.6.** *Let $I \subset S$ be a monomial ideal with $2 \leq g(I) \leq 2n+1$. Then, there exists a $j \in [n]$ such that $t_j(I) \geq g(I) - 2n + 3$.*

*Proof.* We denote $G(I) = \{v_1, \ldots, v_m\}$ and $s(I) = \sum_{i=1}^{m} |supp(v_i)|$. If $m \leq 2n-1$, we are done by Lemma 1.3. Consider $m \in \{2n, 2n+1\}$. By reordering the $v_i$'s we may assume that $x_i|v_i$ and $x_i|v_{n+i}$ for all $i \in [n]$. Since $G(I)$ is a minimal system of generators, we can assume that $|supp(v_{n+i})| \geq 2$ for all $i \in [n]$. Therefore $s(I) \geq \sum_{i=1}^{2n} |supp(v_i)| \geq 3n$. If $m = 2n$, since $s(I) = \sum_{j=1}^{n} t_j(I)$, it follows that there exists a $j \in [n]$ such that $t_j(I) \geq 3 = m - 2n + 3$. If $m = 2n+1$, it follows that $s(I) > 3n$ and therefore there exists a $j \in [n]$ such that $t_j(I) \geq 4 = m - 2n + 3$. □

**Example 1.7.** Let $S := K[x, y]$ and $I \subset S$ be a monomial ideal with $m = g(I)$. We claim that $t_1(I) \geq m - 1$ and $t_2(I) \geq m - 1$. Indeed, if $x_1^a \in G(I)$ for some positive integer $a$, then $t_2(I) = m - 1$, otherwise $t_2(I) = m$. Similarly, if $x_2^b \in G(I)$ for some positive integer $b$, then $t_1(I) = m - 1$, otherwise $t_1(I) = m$.

Let $S := K[x, y, z]$. Let $I = (x_1^3, x_2^3, x_3^3, x_1^2x_2, x_1x_2^2, x_1^2x_3, x_1x_3^2, x_2^2x_3, x_2x_3^2)$. Note that $t_1(I) = t_2(I) = t_3(I) = 5$. We have $m := g(I) = 9$ and $m - 6 + 3 = 6 > 5$. On the other hand, one can easily see that any monomial ideal $I \subset K[x, y, z]$ with $g(I) \leq 8$ satisfies the conclusion of 1.6.

Let $n \geq 4$ be an integer and $S = K[x_1, \ldots, x_n]$. If $n$ is even, we consider the ideal

$$I = (x_1^4, \ldots, x_n^4, x_1^3x_2, x_1x_2^3, x_3^3x_4, x_3x_4^3, \ldots, x_{n-1}^3x_n, x_{n-1}x_n^3, x_1^2x_2^2, x_3^2x_4^2).$$

We have $m := g(I) = 2n+2$ and $t_j(I) \leq 4$ for all $j \in [n]$. Therefore $m - 2n + 3 = 5 > 4$.

If $n$ is odd, we consider the ideal

$$I = (x_1^4, \ldots, x_n^4, x_1^3x_2, x_3^3x_4, \ldots, x_n^3x_1, x_2^3x_3, \ldots, x_{n-1}^3x_n, x_1^2x_2^2, x_3^2x_4^2).$$

We have $m := g(I) = 2n+2$ and $t_j(I) \leq 4$ for all $j \in [n]$. Therefore $m - 2n + 3 = 5 > 4$.

These examples show that the bound for $g(I)$ put in Proposition 1.6 is the best possible, with the exception of the cases $n = 2, 3$.

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